XVII. *General Theorems, chiefly Porisms, in the higher Geometry. By* Henry Brougham, *Jun. Esq. Communicated by Sir* Charles Blagden, *Knt. F. R. S.*

Read May 24, 1798.

The following are a few propositions that have occurred to me, in the course of a considerable degree of attention which I have happened to bestow upon that interesting, though difficult branch of speculative mathematics, the higher geometry. They are all in some degree connected; the greater part refer to the conic hyperbola, as related to a variety of other curves. Almost the whole are of that kind called porisms, whose nature and origin is *now* well known; and, if that mathematician to whom we owe the first distinct and popular account of this formerly mysterious, but most interesting subject,* should chance to peruse these pages, he will find in them additional proofs of the accuracy which characterizes his inquiry into the discovery of this singularly beautiful species of proposition.

Though each of the truths which I have here enunciated is of a very general and extensive nature, yet they are all discovered by the application of certain principles or properties still more general; and are thus only cases of propositions still more extensive. Into a detail of these, I cannot at present enter: they compose a system of general methods, by which the discovery of propositions is effected with certainty and ease; and

* See Mr. Playfair's Paper, in Vol. III. of the Edinburgh Trans.

they are, very probably, in the doctrine of curve lines, what the ancients appear to have prized so much in plain geometry; although, unfortunately, all that remains to us of that treasure, is the knowledge of its high value. Neither have I added the demonstrations, which are all purely geometrical, granting the methods of tangents and quadratures: I have given an example, in the abridged synthesis, of what I consider as one of the most intricate. It is unnecessary to apologize any farther for the conciseness of this tract. Let it be remembered, that were each proposition followed by its analysis and composition, and the corollaries, scholia, limitations, and problems, immediately suggested by it, without any trouble on the reader's part, the whole would form a large volume, in the style of the ancient geometers; containing the investigation of a series of connected truths, in one branch of the mathematics, all arising from varying the combinations of certain data enumerated in a general enunciation.*

As a collection of curious general truths, of a nature, so far as I know, hitherto quite unknown, I am persuaded this Paper, with all its defects, may not be unacceptable to those who feel pleasure in contemplating the varied and beautiful relations between abstract quantities, the wonderful and extensive analogies which every step of our progress in the higher parts of geometry opens to our view.

PROP. I. PORISM. (Tab. XXI. fig. 1.)

A conic hyperbola being given, a point may be found, such, that every straight line drawn from it to the curve, shall cut, in

* See the celebrated account of ancient geometrical works, in the eleventh book of PAPPUS ALEXANDRINUS.

a given ratio, that part of a straight line passing through a given point which is intercepted between a point in the curve not given, but which may be found, and the ordinate to the point where the first mentioned line meets the curve.

Let X be the point to be found, NA the line passing through the given point N, and M any point whatever in the curve; join XM, and draw the ordinate MP; then AC is to CP in a given ratio.

Corollary. This property suggests a very simple and accurate method of describing a conic hyperbola, and then finding its centre, asymptotes, and axes; or, any of these being given, of finding the curve, and the remaining parts.

PROP. II. PORISM.

A conic hyperbola being given, a point may be found, such, that if from it there be drawn straight lines to all the intersections of the given curve, with an infinite number of parabolas, or hyperbolas, of any given order whatever, lying between straight lines, of which one passes through a given point, and the other may be found, the straight lines so drawn, from the point found, shall be tangents to the parabolas, or hyperbolas.

This is in fact two propositions; there being a construction for the case of parabolas, and another for that of hyperbolas.

PROP. III. PORISM.

If, through any point whatever of a given ellipse, a straight line be drawn parallel to the conjugate axis, and cutting the ellipse in another point; and if at the first point a perpendicular be drawn to the parallel, a point may be found, such, that if from it there be drawn straight lines, to the innumerable intersections of the ellipse with all the parabolas of orders not given, but which may be found, lying between the lines drawn at right angles to each other, the lines so drawn from the point found, shall be normals to the parabolas at their intersections with the ellipse.

PROP. IV. PORISM.

A conic hyperbola being given, if through any point thereof a straight line be drawn parallel to the transverse axis, (and cutting the opposite hyperbolas,) a point may be found, such, that if from it there be drawn straight lines, to the innumerable intersections of the given curve with all the hyperbolas of orders to be found, lying between straight lines which may be found, the straight lines so drawn shall be normals to the hyperbolas at the points of section.

Scholium. The two last propositions give an instance of the many curious and elegant analogies between the hyperbola and ellipse; failing, however, when by equating the axes we change the ellipse into a circle.

PROP. V. LOCAL THEOREM. Fig. 2.

If, from a given point A, a straight line DE moves parallel to itself, and another CS, from a given point C, moves along with it round C; and a point I moves along AB, from H, the middle point of AB, with a velocity equal to half the velocity of DE; then, if AP be always taken a third proportional to AS and BC, and through P, with asymptotes D′E′ and AB, a conic hyperbola be described; also, focus I and axis AB, a conic parabola be described, the *radius vector* from C to M, the intersection of the two curves, moving round C, shall describe a given ellipse.

PROP. VI. THEOREM.

A common logarithmic being given, and a point without it, a parabola, hyperbola, and ellipse, may be described, which shall intersect the logarithmic and each other in the same points; the parabola shall cut the logarithmic orthogonally; and, if straight lines be drawn from the given point to the common intersections of the four curves, these lines shall be normals to the logarithmic.

PROP. VII. PORISM.

Two points in a circle being given, (but not in one diameter,) another circle may be described, such, that if from any point thereof to the given points straight lines be drawn, and a line touching the given circle, the tangent shall be a mean proportional between the lines so inflected.

Or, more generally, the square of the tangent shall have a given ratio to the rectangle under the inflected lines.

PROP. VIII. PORISM. Fig. 3.

Two straight lines AB, AP, (not parallel,) being given in position, a conic parabola MN may be found, such, that if, from any point thereof M, a perpendicular MP be drawn to the one of the given lines nearest the curve, and this perpendicular be produced till it meets the other line in B, and if from B a line be drawn to a given point C, MP shall have to PB together with CB, a given ratio.

Scholium. This is a case of a most general enunciation, which gives rise to an infinite variety of the most curious porisms.

PROP. IX. PORISM. Fig. 4.

A conic hyperbola being given, a point may be found, from which if straight lines be drawn to the intersections of the given curve with innumerable parabolas (or hyperbolas) of any given order whatever, lying between perpendiculars which meet in a given point, the lines so drawn shall cut, in a given ratio, all the *areas* of the parabolas (or hyperbolas) contained by the peripheries and co-ordinates to points thereof, found by the innumerable intersections of another conic hyperbola, which may be found.

This comprehends, evidently, two propositions; one for the case of parabolas, the other for that of hyperbolas. In the former it is thus expressed with a figure.

Let EM be the given hyperbola; BA, AC, the perpendiculars meeting in a given point A: a point X may be found, such, that if XM be drawn to any intersection M of EM with any parabola AMN, of any given order whatever, and lying between AB and AC, XM shall cut, in a given ratio, the area AMNP, contained by AMN and AP, PN, co-ordinates to the conic hyperbola FN, which is to be found; thus, the area ARM shall be to the area RMNP in a given ratio.

PROP. X. PORISM.

A conic hyperbola being given, a point may be found, such, that if from it there be drawn straight lines, to the innumerable intersections of the given curve with all the straight lines drawn through a given point in one of the given asymptotes, the first mentioned lines shall cut, in a given ratio, the areas of all the triangles whose bases and altitudes are the co-ordinates to a second conic hyperbola, which may be found, at the points where it cuts the lines drawn from the given point.

PROP. XI. PORISM.

A conic hyperbola being given, a straight line may be found, such, that if another move along it in a given angle, and pass through the intersections of the curve with all the parabolas, (or hyperbolas,) of any given order whatever, lying between straight lines to be found, the moving line shall cut, in a given ratio, the areas of the curves described, contained by the peripheries and co-ordinates to another conic hyperbola, that may be found, at the points where this cuts the curves described.

PROP. XII. PORISM.

A conic hyperbola being given, a straight line may be found, along which if another move in a given angle, and pass through any point whatever of the hyperbola, and if this point of section be joined with another that may be found, the moving line shall cut, in a given ratio, the triangles whose bases and altitudes are the co-ordinates to a conic hyperbola, which may be found, at the points where it meets the lines drawn from the point found.

Scholium. I proceed to give one or two examples, wherein areas are cut in a given ratio, not by straight lines, but by curves.

PROP. XIII. PORISM. Fig. 5.

A conic hyperbola being given, if through any of its innumerable intersections with all the parabolas of any order, lying between straight lines, whereof one is an asymptote, and the other may be found, an hyperbola of any order be described, (except the conic,) from a given origin in the given asymptote perpendicular to the axis of the parabolas, the hyperbola thus described shall cut, in a given ratio, an area (of the parabolas) which may be always found.

If from G (as origin) in AB, one of LM's asymptotes, there be described an hyperbola IC′, of any order whatever, except the first, and passing through M, a point where LM cuts any of the parabolas AM, of any order whatever, drawn from A a point to be found, and lying between AB and AC, an area ACD may be always found, (that is, for every case of AM and IC′,) which shall be constantly cut by IC′, in the given ratio of M : N; that is, the area AMN : NMDC : : M : N.

I omit the analysis, which leads to the following construction and composition.

Construction. Let $\overline{m+n}$ be the order of the parabolas, and $\overline{p+q}$ that of the hyperbolas. Find φ a fourth proportional to $m+n$, $\overline{q-p}$ and $m+2n$; divide GB in A, so that AR : AG : : q : $p+\varphi$; then find π a fourth proportional to M + N, $\varphi+p$, and $q-p$, and γ a fourth proportional to q, AG, and $q-p$; and, lastly, θ a fourth proportional to the parameter* of LM, π and M. If, with a parameter equal to $\frac{\overline{m+n}}{m+2n} \times \theta - \frac{M+N}{M}$ of the rectangle τ . AN, and between the asymptotes AB, AC, a conic hyperbola be described, it shall cut the parabola in a point, the co-ordinates to which contain an area that shall be cut by IC′ in the ratio of M : N.

Demonstration. Because AG is divided in R, so that AR : AG

* *i. e.* The constant rectangle or space to which AP . SM is equal.

$:: q : p + \phi$, and that $\phi : m + n :: q - p : m + 2n$, AR is equal to $\frac{AG \times q}{p + \frac{\overline{m+n} \times \overline{q-p}}{m+2n}}$; and, because LM is a conic hyperbola, the rectangle MS . RS, or MS . AP, or AP . $\overline{MP + AR}$ is equal to the parameter, (or constant space,) therefore, this parameter is equal to AP $\times$ $\overline{MP + \frac{AG \cdot q}{p + \frac{(m+n)(q-p)}{m+2n}}}$

Again, the space ACD is equal to $\frac{m+n}{m+2n}$ of the rectangle AC . CD, since AD is a parabola of the order $m + n$; but (by construction) AC . CD is equal to $\frac{m+n}{m+2n}$ of $\overline{\theta - \frac{M+N}{M} \cdot \tau \cdot AN}$; therefore, ACD $= \theta - \frac{M+N}{M} \cdot \tau \cdot AN$, of which θ : parameter of LM $:: \pi : M$, and $\pi : M + N :: \phi + p : q - p$; therefore, $\theta = \frac{\text{Par. LM} \times \overline{M+N}}{M\,(q-p)} \times \left(\frac{\overline{m+n} \times \overline{q-p}}{m+2n} + p;\right)$ also, $\tau : q :: AG : q - p$; consequently, ACD $= \frac{\text{Par. LM} \times \overline{M+N}}{M\,(q-p)}$ multiplied by $\left(\frac{\overline{m+n} \times \overline{q-p}}{m+2n} + p\right)$ and diminished by $\frac{M+N}{M} \times AN \times \frac{q \cdot AG}{q-p}$; therefore, transposing $\frac{\text{Par. LM} \times \overline{M+N}}{M \times \overline{q-p}} \times \left(\frac{m+n}{m+2n} \times \overline{q-p} + p\right)$ is equal to ACD $+ \frac{\overline{M+N}}{M} \times AN \times \frac{q \cdot AG}{q-p}$; and Par. LM will be equal to $\frac{\left(\overline{ACD + \frac{M+N}{M} \times AN \times \frac{q \cdot AG}{q-p}}\right) \times \frac{M}{q-p}}{\left(\frac{\overline{m+n}}{m+2n} \times \overline{q-p} + p\right) \times \overline{M+N}}$, that is,

$$\frac{\frac{M}{M+N} \times \overline{q-p} \times ACD + q \cdot AN \times AG}{\frac{m+n}{m+2n} \times \overline{q-p} + p}.$$

Now, it was before demonstrated, that the parameter of LM is equal to $\text{AP} \times \dfrac{\overline{\text{MP} + q\,.\,\text{AG}}}{p + \frac{(m+n)(q-p)}{m+2n}}$. This is therefore equal to $\dfrac{\frac{\text{M}}{\text{M}+\text{N}} \times \overline{q-p} \times \text{ACD} + q\,.\,\text{AN} \times \text{AG}}{\frac{m+n}{m+2n} \times \overline{q-p} + p}$, multiplying both by $\dfrac{\overline{m+n} \times \overline{q-p}}{m+2n} + p$, we have $\frac{\text{M}}{\text{M}+\text{N}} \times \overline{q-p} \times \text{ACD} + q\,.\,\text{AN} \times \text{AG}$ $= \text{AP} \times \left(\overline{\text{MP} \times \left(p + \frac{\overline{m+n} \times \overline{q-p}}{m+2n}\right) + q\,.\,\text{AG}}\right)$.

From these equals take $q\,.\,\text{AG} \times \text{AN}$, and there remains $\frac{\text{M}}{\text{M}+\text{N}} \times \overline{q-p} \times \text{ACD}$ equal to $\text{AP} \times \text{PM} \times \left(\frac{\overline{m+n} \times \overline{q-p}}{m+2n} + p\right)$ $+ q\,.\,\text{AG} \times \overline{\text{AP} - \text{AN}}$; or, dividing by $q - p$; $\frac{\text{M}}{\text{M}+\text{N}} \times \text{ACD}$ $= \text{AP} \times \overline{\frac{m+n}{m+2n} + \frac{p}{q-p}} \times \text{PM} + \frac{q}{q-p} \times \text{AG} \times \overline{\text{AP} - \text{AN}}$. Now, $\frac{m+n}{m+2n} \times \text{AP} \times \text{PM}$ is equal to the area APM; therefore, the area APM together with $\frac{p}{q-p} \times \text{AP}.\,\text{PM}$, and $\frac{q}{q-p} \times \text{AG} \times \overline{\text{AP} - \text{AN}}$, or APM with $\frac{p}{q-p} \times \text{AP}.\,\text{PM} - \frac{q}{q-p} \times \text{AG} \times \overline{\text{AN} - \text{AP}}$, or $\text{APM} + \frac{q}{q-p} \times \text{AP}\,.\,\text{PM} - \frac{q}{q-p} \times$ rect. PT is equal to $\frac{\text{M}}{\text{M}+\text{N}}$ $\times$ACD. Now, IC′ is an hyperbola of the order $p + q$; therefore, its area is $\frac{p}{p-q} \times$ rect. GH . MH. But q is greater than p; therefore, $\frac{p}{p-q}$ is negative, and $\frac{p \times \overline{\text{GH}\,.\,\text{HM}}}{q-p}$ is the area MHKC′; and the area NTKC′ is equal to $\frac{p}{q-p} \times \text{GT} \times \text{TN}$; therefore, MNTH is equal to (MHKC′—NTKC′), or to $\frac{p}{q-p} \times \overline{\text{GH}.\text{MH} - \text{GT}.\text{TN}}$. From these equals take the common rectangle AT, and there remains the area MPN, equal to $\frac{p}{q-p} \times \text{AP} \times \text{MP} - \frac{q}{q-p} \times \text{PT}$; which was before demonstrated to be, together with APM, equal to $\frac{\text{M}}{\text{M}+\text{N}} \cdot \text{ACD}$. Wherefore MPN, together with APM, that is, the area AMN is equal to $\frac{\text{M}}{\text{M}+\text{N}} \cdot \text{ACD}$; consequently, AMN : ACD :: M : M + N; and (*dividendo*) AMN : NMDC :: M : N. An area has therefore been found, which the hyperbola IC′ always cuts in a given ratio.

Wherefore, a conic hyperbola being given, *&c. Q. E. D.*

Scholium. This proposition points out, in a very striking manner, the connection between all parabolas and hyperbolas, and their common connection with the conic hyperbola. The demonstration which I have given is much abridged; and, to avoid circumlocution, algebraic symbols, and even ideas, have been introduced: but, by attending to the several steps, any one will easily perceive that it may be translated into geometrical language, and conducted upon purely geometrical principles, if any numbers be substituted for m, n, p, and q; or if these letters be made representatives of lines, and if conciseness be less rigidly studied.

PROP. XIV. THEOREM.

A common logarithmic being given, if from a given point, as origin, a parabola (or hyperbola) of any order whatever be described, cutting, in a given ratio, a given area of the logarithmic, the point where this curve meets the logarithmic is always situated in a conic hyperbola, which may be found.

Scholium. This proposition is, properly speaking, neither a porism, a theorem, nor a problem. It is not a theorem, because something is left to be found, or, as PAPPUS expresses it, there is a deficiency in the hypothesis: neither is it a porism; for the theorem, from which the deficiency distinguishes it, is not local.

PROP. XV. PORISM. Fig. 6.

A conic hyperbola being given, two points may be found, from which if straight lines be inflected, to the innumerable intersections of the given curve with parabolas or hyperbolas, of any given order whatever, described between given straight lines, and if co-ordinates be drawn to the intersections of these curves with another conic hyperbola, which may be found, the lines inflected shall always cut off areas that have to one another a given ratio, from the areas contained by the co-ordinates.

Let X and Y be the points found; HD the given hyperbola, FE the one to be found; ADC one of the curves lying between AB and AG, intersecting HD and FE: join XD, YD; then the area AYD : XDCB in a given ratio.

PROP. XVI. PORISM. Fig. 7.

If, between two straight lines making a right angle, an infinite number of parabolas, of any order whatever, be described, a conic parabola may be drawn, such, that if tangents be drawn to it at its intersections with the given curves, these tangents shall always cut, in a given ratio, the areas contained by the given curves, the curve found, and the axis of the given curves.

Let AMN be one of the given parabolas; DMO the parabola found, and TM its tangent at M: ATM shall have to TMR a given ratio.

PROP. XVII. PORISM.

A parabola of any order being given, two straight lines may be found, between which if innumerable hyperbolas of any order be described, the areas cut off by the hyperbolas and the given parabola at their intersections, shall be divided, in a given ratio, by the tangents to the given curve at the intersections; and, conversely, if the hyperbolas be given, a parabola may be found, *&c. &c.*

PROP. XVIII. PORISM.

A parabola of any order $(m + n)$ being given, another of an order $(m + 2n)$ may be found, such, that the rectangle under its ordinate and a given line, shall have always a given ratio to the area (of the given curve) whose abscissa bears to that of the curve found a given ratio.

Example. Let $m = 1$, $n = 1$, and let the given ratios be those of equality; the proposition is this; a conic parabola being given, a semicubic one may be found, such, that the rectangle under its ordinate and a given line, shall be always equal to the area of the given conic parabola, at equal abscissæ.

Scholium. A similar general proposition may be enunciated and exemplified, with respect to hyperbolas; and, as these are only cases of a proposition applying to all curves whatsoever, I shall take this opportunity of introducing a very simple, and, I think, perfectly conclusive demonstration, of the 28th lemma, *Principia*, Book I. "*that no oval can be squared.*" It is well known, that the demonstration which Sir ISAAC NEWTON gives of this lemma, is not a little intricate; and, whether from this difficulty, or from some real imperfection, or from a very natural wish not to believe that the most celebrated *desideratum* in geometry must for ever remain a *desideratum*, certain it is, that many have been inclined to call in question the conclusiveness of that proof.

Let AMC be any curve whatever, (fig. 8.) and D a given line; take in *a b* a part *ap*, having to AP a given ratio, and erect a perpendicular *p m*, such, that the rectangle *pm* . D shall have to the area APM a given ratio; it is evident that *m* will describe a curve *amc*, which can never cut the axis, unless in *a*. Now, because *pm* is proportional to $\frac{APM}{D}$, or to APM, *pm* will always increase, *ad infinitum*, if AMC is infinite; but, if AMC stops or returns into itself, that is, if it is an oval, *pm* is a *maximum* at *b*, the point of *ab* corresponding to B in AB; consequently, the curve *amc stops short*, and is *irrational*. Therefore *pm*, its ordinate, has not a finite relation to *ap*, its abscissa; But *ap* has a given ratio to AP; therefore *pm* has not a finite relation to AP, and APM has a given ratio to *pm*; therefore it has not a finite relation to AP, that is, APM cannot be found in finite terms of AP, or is incommensurate with AP; wherefore, the curve AMB cannot be squared. Now, AMB is any oval; wherefore no oval can be squared. By an argument of precisely the same kind, it may proved, that the *rectification*, also, of every oval is impossible. Wherefore, *&c. Q. E. D.*

I shall subjoin three problems, that occurred during the consideration of the foregoing propositions. The first is an example of the application of the porisms to the solution of problems. The second gives, besides, a new method of resolving one of the most celebrated ever proposed, KEPLER's problem; and the last presents to our view a curve before unknown, (at least to me,) as possessing the singular property of a constant tangent.

PROP. XIX. PROBLEM. Fig. 9.

A common logarithmic being given, to describe a conic hyperbola, such, that if from its intersection with the given curve a straight line be drawn to a given point, it shall cut a given area of the logarithmic in a given ratio. The analysis leads to this

Construction. Let BME be the logarithmic, G its modula; AB the ordinate at its origin A; let C be the given point; ANOB the given area; M : N the given ratio: draw BQ parallel to AN; find D a fourth proportional to M, the rectangle BQ . OQ, and M + N. From AD cut off a part AL, equal to AC together with twice G; at L, make LH perpendicular to AD, and, between the asymptotes AL, HL, with a parameter * twice $(\overline{D + 2 . AB . G})$ describe a conic hyperbola: it is the curve required.

PROP. XX. PROBLEM. Fig. 10.

To draw through the focus of a given ellipse, a straight line that shall cut the area of the ellipse in a given ratio.

Construction. Let AB be the transverse axis, EF the semiconjugate; E, of consequence, the centre; C and L the foci. On AB describe a semicircle. Divide the quadrant AK in the given ratio in which the area is to be cut, and describe the cycloid GMR, such, that the ordinate PM may be always a

* Or constant rectangle.

fourth proportional to the arc OQ, the rectangle AB × 2 . FE, and the line CL; this cycloid shall cut the ellipse in M, so that, if MC be joined, the area ACM shall be to CMB : : M : N.

Demonstration. Let $AP = x$, $PM = y$, $AC = c$, $AB = a$, and $2 . EF = b$; then, by the nature of the cycloid GMR, $-PM : \overline{OQ} :: 2 . FE \times AB : cL$, and $QO = AO - AQ =$ (by const.) $\frac{M}{M+N} \times \overline{AK - AQ}$; also, $CL = AB - 2 . AC$, since $AC = LB$. Therefore, $-PM : \frac{M}{M+N} \times AK - AQ :: AB \times 2 . EF : AB - 2 . AC$; or $-y : \frac{M}{M+N} \times \text{arc} . 90^\circ - \text{arc.vers.sin.}\, x :: ab : a - 2c$; therefore, $-y(a-2c)$ or $+y(2c-a) = ab \times \overline{\left(\frac{M}{M+N} \times \text{arc} . 90^\circ - \text{arc.v.s.}\, x\right)}$ and, by transposition,

$ab \times \text{arc} . \overline{\text{v. s.}}\, x + y(2c - a) = \frac{ab . M}{M+N} \times \text{arc} . 90^\circ$. To these equals, add $2y(x - x) = o$, and multiply by -1; then will $ab \times \text{arc}.\overline{\text{v.s.}}\, x + (2x - a)y - 2y(x - c) = \frac{M}{M+N} \times ab \times \text{arc}. 90^\circ$, of which the fourth parts are also equal; therefore,

$$\frac{ab \times \text{arc}.\overline{\text{v.s.}}\, x}{4} + \frac{(2x-a)y}{4} - \frac{y}{2}(x - c) = \frac{ab}{4} \times \frac{M}{M+N} \times \text{arc} . 90^\circ.$$

Now, because AFB is an ellipse, $y^2 = \frac{b^2}{a^2} \times \overline{ax - x^2}$, and $y = \frac{b}{a}\sqrt{ax - x^2}$; therefore, $\frac{ab \times \text{arc}.\overline{\text{v.s.}}\, x}{4} + \frac{(2x-a)}{4} \times \frac{b}{a}\sqrt{ax - x^2} - \frac{y}{2}(x - c) = \frac{ab}{4} \times \frac{M}{M+N} \times \text{arc} . 90^\circ$. Multiply both numerator and denominator of the first and last terms by a; then will $\frac{b}{a} \times \frac{a^2}{4} \times \text{arc} . \overline{\text{v.s.}}\, x + \frac{2x - a}{4} \times \frac{b}{a}\sqrt{ax - x^2} - \frac{y}{2}(x - c) = \frac{b}{a} \times \frac{a^2}{4} \times \frac{M}{M+N} \times \text{arc} . 90^\circ$. Now, the fluxion of an arc whose versed sine is x and radius $\frac{a}{2}$, is equal to $\frac{a\dot{x}}{2\sqrt{ax - x^2}}$, which is also the fluxion of the arc whose sine is $\sqrt{\frac{x}{a}}$ and radius unity;* wherefore,

* The semi-transverse is supposed unity, through this demonstration.

$\frac{b}{a} \times \left(\frac{a^2}{4} \times \text{arc . sin.} \sqrt{\frac{x}{a}} + \frac{\overline{2x - a}}{4} \times \sqrt{ax - x^2}\right) - \frac{y}{2}(x - c)$ is equal to $\frac{b}{a} \times \frac{a}{4} \times \frac{M}{M + N} \times \text{arc . } 90^\circ$; and, by the quadrature of the circle, $\frac{a^2}{4} \times \text{arc . sin.} \sqrt{\frac{x}{a}} + \frac{\overline{2x - a}}{4} \times \sqrt{ax - x^2}$, is the area whose abscissa is x; consequently, the semicircle's area is $\frac{a^2}{4} \times \text{arc . } 90^\circ$. But the areas of ellipses are, to the corresponding areas of the circles described on their transverse axes, as the conjugate to the transverse; therefore $\frac{b}{a} \times \left(\frac{a^2}{4} \times \text{arc . sin.} \sqrt{\frac{x}{a}} + \frac{\overline{2x - a}}{4} \times \sqrt{ax - x^2}\right)$ is the area whose abscissa is x, of a semi-ellipse whose axes are a and b; and, consequently, $\frac{b}{a} \times \frac{a^2}{4} \times \text{arc . } 90^\circ$ is the area of the semi-ellipse. Wherefore, the area APM $- \frac{y}{2}(x - c)$ is equal to $\frac{M}{M + N}$ of AMFB. But $\frac{y}{2}(x - c) \left(= \frac{PM}{2} \times \overline{AP - AC} = \frac{PM^2}{2} \times PC\right)$ is the triangle CPM; consequently, APM — CPM, or ACM, is equal to $\frac{M}{M+N}$ $\times$ AMFB; and ACM : AMFB :: M : M + N; or (*dividendo*) ACM : CMFB :: M : N; and the area of the ellipse is cut in a given ratio by the line drawn through the focus. *Q. E. D.*

Of this solution it may be remarked, that it does not assume as a postulate the description of the cycloid, but gives a simple construction of that curve, flowing from a curious property, whereby it is related to a given circle. This cycloid too gives, by its intersection with the ellipse, the point required, directly, and not by a subsequent construction, as Sir I. NEWTON's does. I was induced to give the demonstration, from a conviction that it is a good instance of the superiority of modern over ancient analysis; and in itself, perhaps, no inelegant specimen of algebraic demonstration.

PROP. XXI. PROBLEM. Fig. 11.

To find the Curve whose Tangent is always of the same Magnitude.

Analysis. Let MN be the curve required, AB the given axis, SM a tangent at any point M, and let d be the given magnitude; then, $SM.q. = SP.q. + PM.q. = d^2$; or, $y^2 + \frac{y^2 \dot{x}^2}{\dot{y}^2} = d^2$, and $\frac{\dot{x}^2}{\dot{y}^2} = \frac{d^2 - y^2}{y^2}$; therefore, $\dot{x} = \frac{\dot{y}}{y} \times \sqrt{d^2 - y^2}$. In order to integrate this equation, divide $\frac{\dot{y}}{y}\sqrt{d^2 - y^2}$ into its two parts, $\frac{d^2 \dot{y}}{y\sqrt{d^2 - y^2}}$ and $\frac{-y\dot{y}}{\sqrt{d^2 - y^2}}$.

To find the fluent of the former,

$$\frac{d^2 \dot{y}}{y\sqrt{d^2 - y^2}} = \frac{d^2 \dot{y}}{y} \times \frac{\left(1 + \frac{d}{\sqrt{d^2 - y^2}}\right)}{d + \sqrt{d^2 - y^2}} = \frac{-d \times \left(-\frac{d\dot{y}}{y^2} - \frac{d^2 \dot{y}}{y^2\sqrt{d^2 - y^2}}\right)}{\frac{d + \sqrt{d^2 - y^2}}{y}}$$

$$= -\frac{d \times \text{fluxion of } \frac{d + \sqrt{d^2 - y^2}}{y}}{\frac{d + \sqrt{d^2 - y^2}}{y}}$$; therefore, the fluent of $\frac{d^2 \dot{y}}{y\sqrt{d^2 - y^2}}$ is $-d \times$ hyp. log. $\frac{d + \sqrt{d^2 - y^2}}{y}$, and the fluent of the other part, $\frac{-y\dot{y}}{\sqrt{d^2 - y^2}}$ is $+\sqrt{d^2 - y^2}$; therefore, the fluent of the aggregate $\frac{\dot{y}}{y}\sqrt{d^2 - y^2}$ is $\sqrt{d^2 - y^2} - d \times$ h. l. $\frac{d + \sqrt{d^2 - y^2}}{y}$, or $\sqrt{d^2 - y^2} + d \times$ h. l. $\frac{y}{d + \sqrt{d^2 - y^2}}$; a final equation to the curve required.

Q. E. I.

I shall throw together, in a few corollaries, the most remarkable things that have occurred to me concerning this curve.

Corollary 1. The subtangent of this curve is $\sqrt{d^2 - y^2}$.

Corollary 2. In order to draw a tangent to it, from a given point without it; from this point as pole, with radius equal to d, and the curve's axis as directrix, describe a *conchoid of Nicomedes:* to its intersections with the given curve draw straight lines from the given point; these will touch the curve.

Corollary 3. This curve may be described (organically) by drawing one end of a given flexible line or thread along a straight line, whilst the other end is urged by a weight towards the same straight line. It is, consequently, the *curve of traction* to a straight line.

Corollary 4. In order to describe this curve from its equation; change the one given above, by *transferring* the axes of its co-ordinates: it becomes (y being $=$ P′M and $x =$ AP′) $y = \sqrt{d^2 - x^2} + d \times \text{h. l.} \frac{x}{d + \sqrt{d^2 - x^2}}$; which may be used with ease, by changing the hyperbolic into the tabular logarithm.

Thus then, the common logarithmic has its *subtangent* constant; the conic parabola, its *subnormal;* the circle, its *normal;* and the curve which I have described in this proposition, its *tangent.*

Edinburgh, 1797.

Fig. 1.

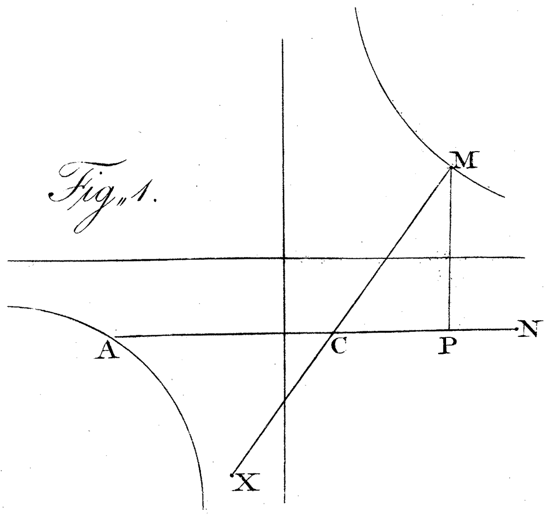
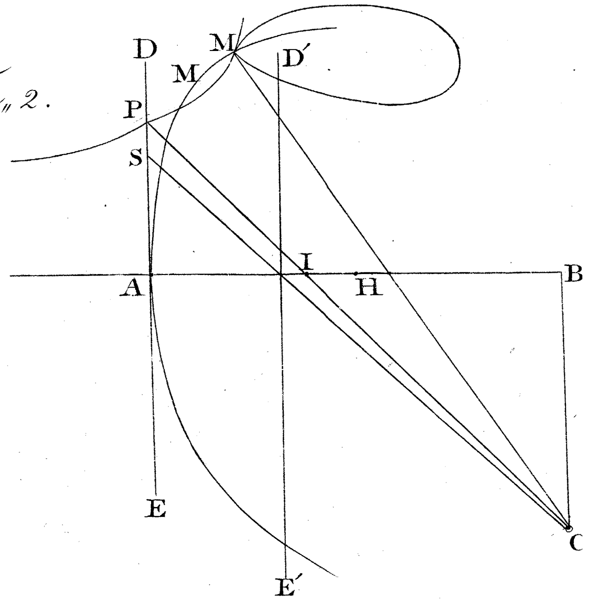


Fig. 2.



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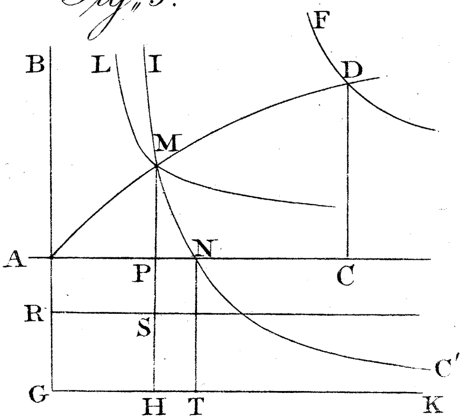

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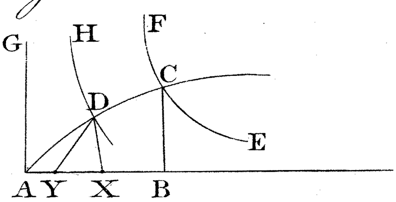

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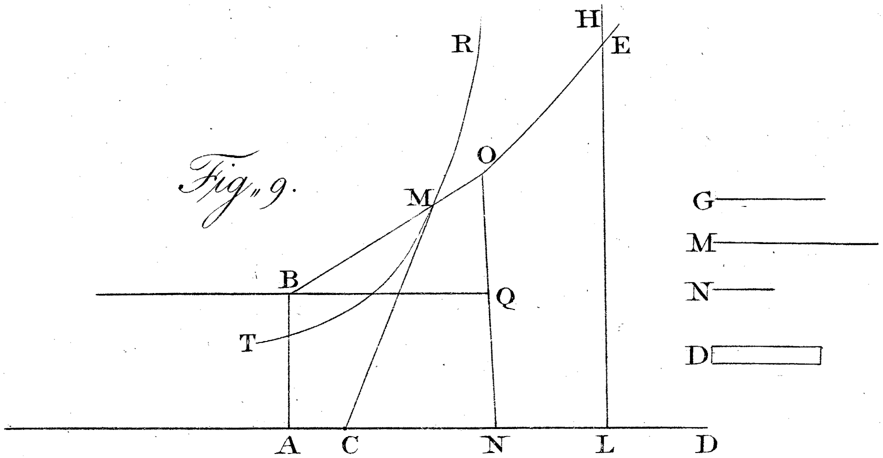

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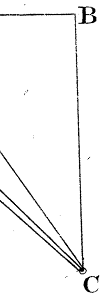

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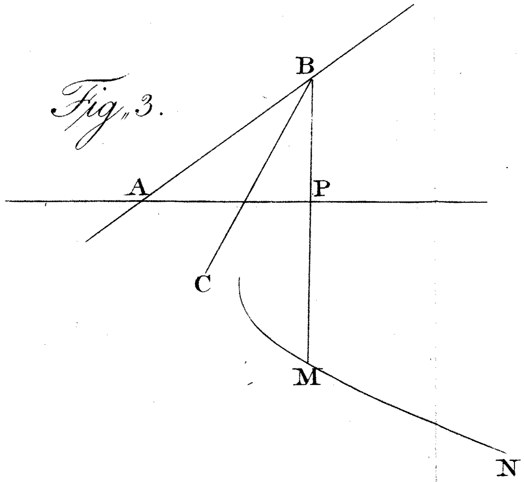

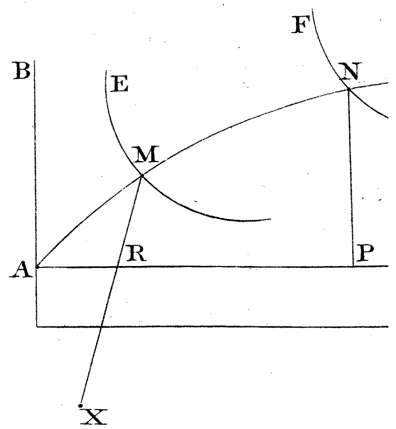

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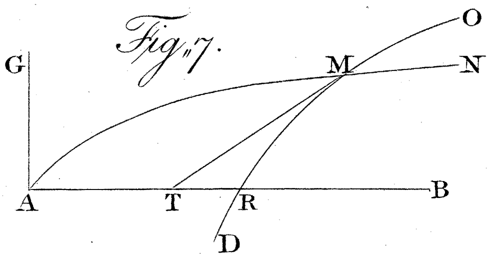

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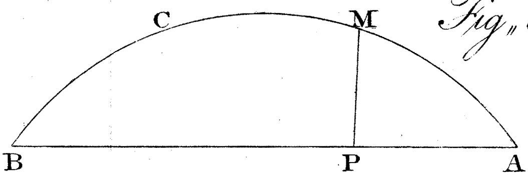
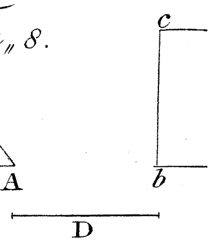
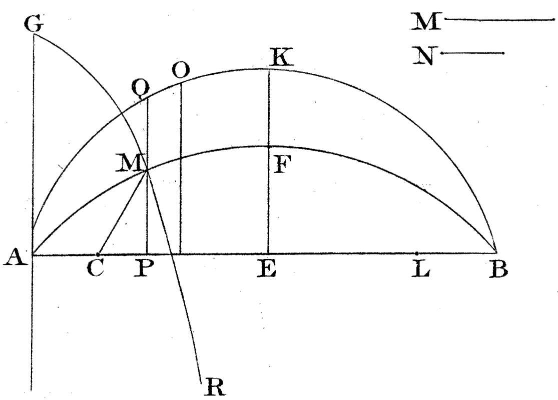






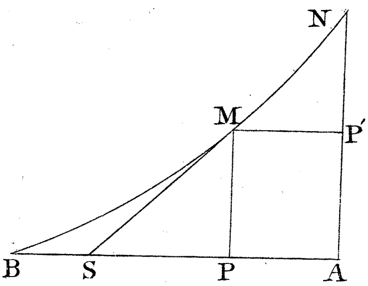
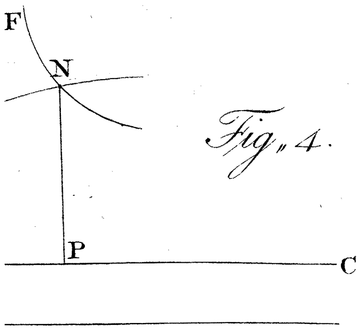


Fig. 11



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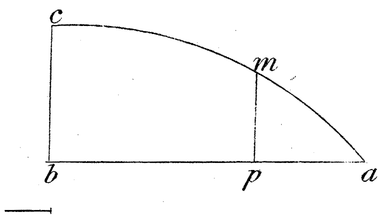

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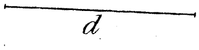

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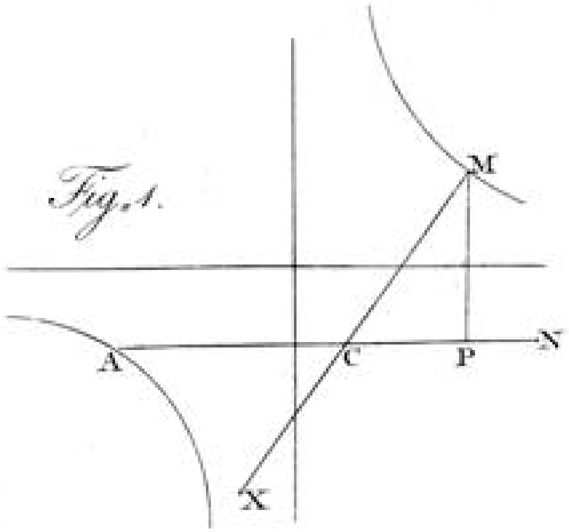

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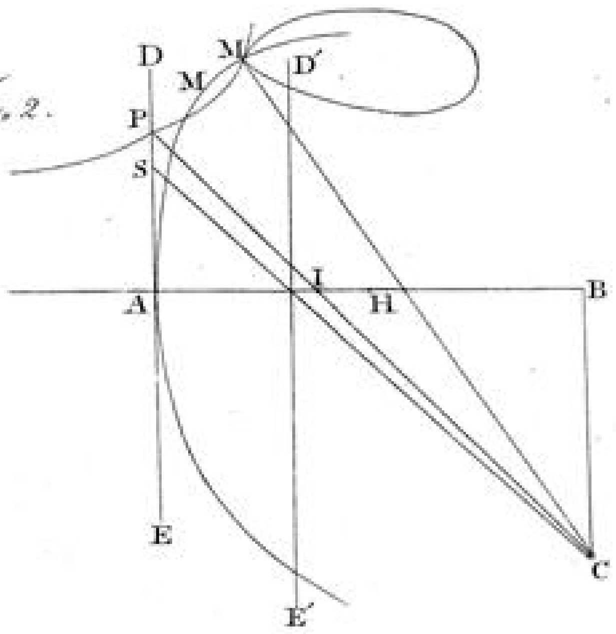

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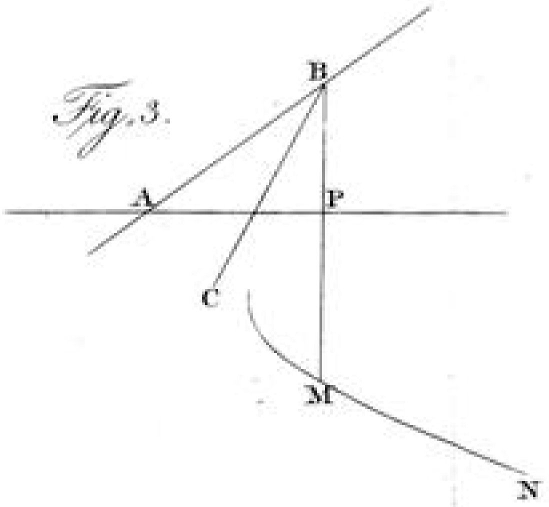

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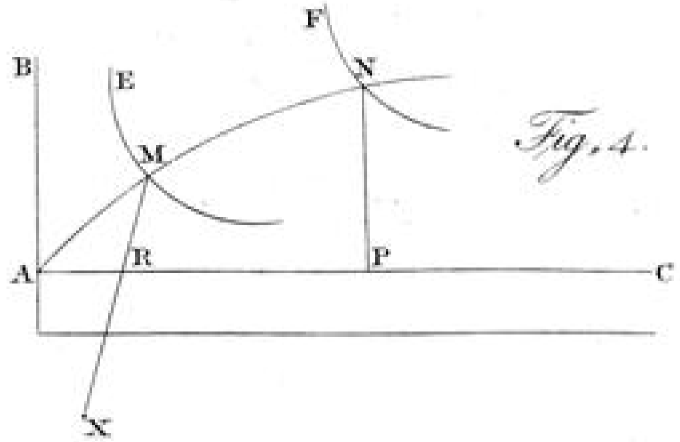

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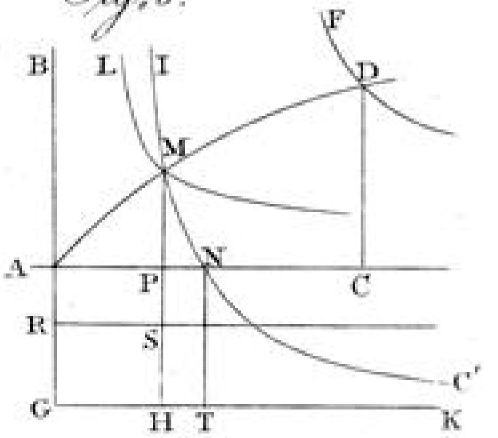

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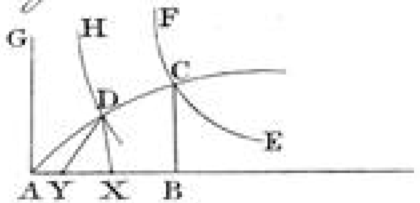

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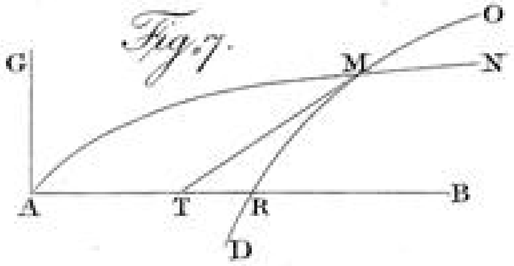

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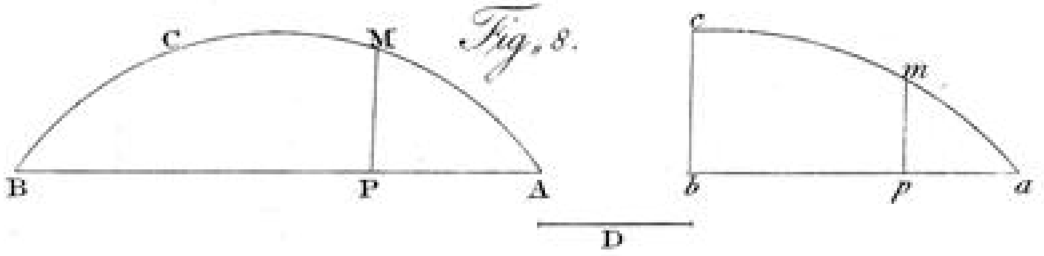

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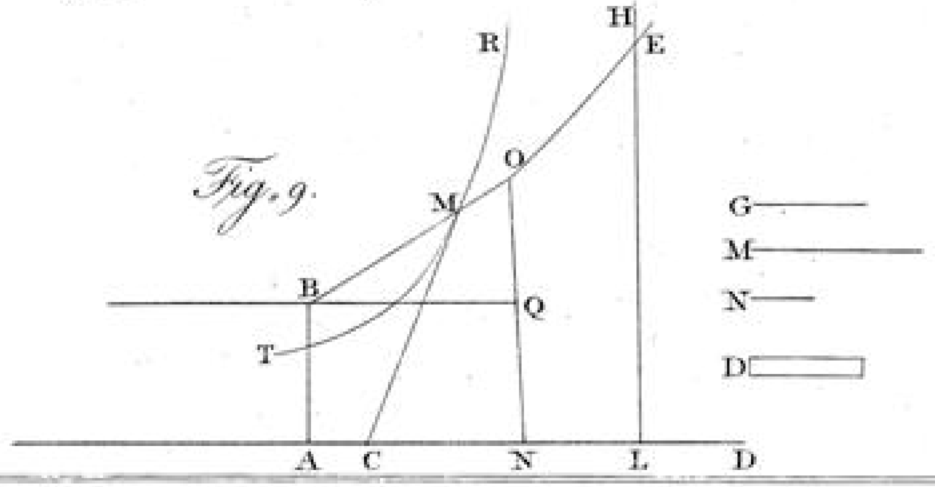

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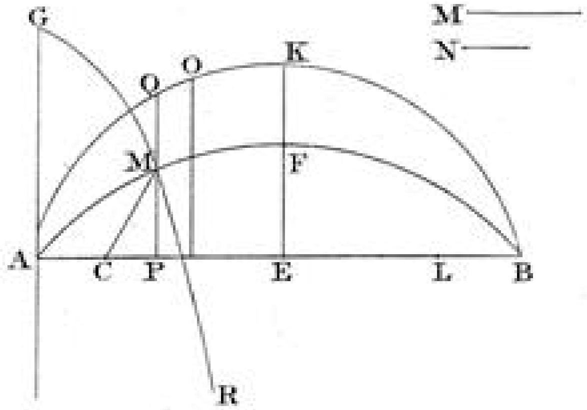

Fig. 11.